

OPERATOR SPACE AND OPERATOR SYSTEM ANALOGS OF KIRCHBERG'S NUCLEAR EMBEDDING THEOREM

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ABSTRACT. The Gurarij operator space \mathbb{NG} introduced by Oikhberg is the unique separable 1-exact operator space that is approximately injective in the category of 1-exact operator spaces and completely isometric linear maps. We prove that a separable operator space X is nuclear if and only if there exist a linear complete isometry $\varphi : X \rightarrow \mathbb{NG}$ and a completely contractive projection from \mathbb{NG} onto the range of φ . This can be seen as the operator space analog of Kirchberg's nuclear embedding theorem. We also establish the natural operator system analog of Kirchberg's nuclear embedding theorem involving the Gurarij operator system \mathbb{GS} .

1. INTRODUCTION

Nuclearity and exactness are properties of fundamental importance for the theory of C^* -algebras and the classification program. The celebrated Kirchberg exact embedding theorem characterizes (up to $*$ -isomorphism) the separable exact C^* -algebras as the C^* -subalgebras of the Cuntz algebra \mathcal{O}_2 [RS, Theorem 6.3.11]; see also [K1, KP]. Furthermore the Kirchberg nuclear embedding theorem asserts that (up to $*$ -isomorphism) the separable nuclear C^* -algebras are precisely the C^* -subalgebras of \mathcal{O}_2 that are the range of a (completely) contractive projection [RS, Theorem 6.3.12]. The Cuntz algebra \mathcal{O}_2 , initially introduced and studied in [C], is the C^* -algebra generated by two isometries of the Hilbert space with orthogonally complementary ranges.

The main result of this paper is an analog of Kirchberg's nuclear embedding theorem for operator spaces involving the (*noncommutative*) Gurarij operator space. This is the unique separable 1-exact operator space \mathbb{NG} which is approximately injective in the category of 1-exact operator spaces with completely isometric maps. In other words \mathbb{NG} is characterized by the following property: for any $n \in \mathbb{N}$, $\varepsilon > 0$, finite-dimensional 1-exact operator spaces $E \subset F$, and complete isometry $\phi : E \rightarrow \mathbb{NG}$, there exists a

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complete isometry $\psi : F \rightarrow \mathbb{N}\mathbb{G}$ such that $\|\psi|_E - \phi\| \leq \varepsilon$. Such a space can be thought as a noncommutative analog of the Gurarij Banach space \mathbb{G} , which is defined by the same property above where one considers Banach spaces instead of 1-exact operator spaces [G,L4,KS].

Existence of the Gurarij operator space was first proved in [O], while uniqueness was established in [L2]. It is worth mentioning here that, albeit being 1-exact, $\mathbb{N}\mathbb{G}$ does not admit any completely isometric embedding into an exact C*-algebra by [L3, Corollary 4.16]. In particular $\mathbb{N}\mathbb{G}$ is not completely isometric to a C*-algebra.

A separable operator space is 1-exact if and only if it admits a completely isometric embedding into $\mathbb{N}\mathbb{G}$ [L2, Theorem 4.3]. (Modulo uniqueness of $\mathbb{N}\mathbb{G}$, this also follows from [O, Theorem 1.1] and the proof of [EOR, Theorem 4.7].) Such a result can be regarded as an operator space version of Kirchberg's exact embedding theorem. In this paper we prove the natural analog of Kirchberg's nuclear embedding theorem for operator spaces: the separable nuclear operator spaces are (up to completely isometric isomorphism) precisely the subspaces of $\mathbb{N}\mathbb{G}$ that are the range of a completely contractive projection; see Theorem 2.5.

We also observe that all the results mentioned above hold for operator systems, as long as one only considers *unital* linear maps, and replaces the Gurarij operator space $\mathbb{N}\mathbb{G}$ with the Gurarij operator system $\mathbb{G}\mathbb{S}$. Such an operator system is uniquely characterized among separable 1-exact operator systems similarly as $\mathbb{N}\mathbb{G}$, where one considers complete order embeddings instead of complete isometries. Existence and uniqueness of such an operator system, as well as universality among separable 1-exact operator systems, have been established in [L3]. The analog of Kirchberg's nuclear embedding theorem in this context asserts that the separable 1-exact operator systems are (up to complete order isomorphism) precisely the subsystems of $\mathbb{G}\mathbb{S}$ that are the range of a unital completely positive projection; see Theorem 3.3.

It is worth mentioning that it should come as no surprise that, while the Kirchberg's embedding theorems involve the Cuntz algebra \mathcal{O}_2 , their operator space and operator system versions involve the Gurarij operator space and operator system. In fact by [KP, Theorem 1.13 and Theorem 2.8] any separable exact C*-algebra embeds into \mathcal{O}_2 , and any two unital embeddings of a separable unital exact C*-algebra into \mathcal{O}_2 are approximately unitarily equivalent. It therefore follows from [BY, Theorem 2.21] that finitely generated exact unital C*-algebras form a Fraïssé class with limit \mathcal{O}_2 . The Gurarij operator space and operator systems can be similarly described as the Fraïssé limits in the sense of [BY] of the classes of finitely-generated 1-exact operator spaces and operator systems; see [L2,L3]. It is therefore natural to expect that they exhibit similar properties as \mathcal{O}_2 in the respective categories.

The present paper is organized as follows. In Section 2 we prove the above mentioned results concerning operator spaces. We recall a few basic notions about operator spaces in Subsection 2.1. Subsection 2.2 contains

canonical approximate amalgamation results for operator spaces, while a characterization of nuclearity is recalled in Subsection 2.3. The operator space analog of Kirchberg's nuclear embedding theorem is then proved in Subsection 2.4. The analogous results for operator systems are stated in Section 3. The proofs are similar, and only the relevant changes to be made from the operator space case will be pointed out.

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2. NUCLEAR OPERATOR SPACES

2.1. Some background notions on operator spaces. An *operator space* X is a closed subspace of the algebra $B(H)$ of bounded linear operators on a complex Hilbert space H . The identification of the space $M_n(X)$ of $n \times n$ matrices with entries in X with a space of operators on the n -fold Hilbertian sum $H^{\oplus n}$ induces, for every $n \in \mathbb{N}$, a norm $M_n(X)$ (the operator norm). The *matricially normed* vector spaces that arise in this way have been characterized by Ruan [R, Theorem 3.1] as those satisfying the following condition:

$$\left\| \sum_{i=1}^m \alpha_i^* x_i \beta_i \right\| \leq \left\| \sum_{i=1}^m \alpha_i^* \alpha_i \right\| \max_{1 \leq i \leq m} \|x_i\| \left\| \sum_{i=1}^m \beta_i^* \beta_i \right\| \quad (1)$$

where $m \in \mathbb{N}$, $\alpha_i, \beta_i \in M_{k \times n}$ (the space of complex $k \times n$ matrices), $x_i \in M_k(X)$, the norm of scalar matrices is the operator norm, and $\alpha_i^* x_i \beta_i$ denotes the row-column matrix multiplication. The identification of M_n with $B(H)$ where H is the n -dimensional Hilbert space induces a canonical operator space structure on the space M_n of $n \times n$ complex matrices.

If $\varphi : X \rightarrow Y$ is a linear map between operator spaces, its n -th *amplification* is the map $\varphi^{(n)} : M_n(X) \rightarrow M_n(Y)$ obtained by applying φ entrywise. We set $\|\varphi\|_n := \|\varphi^{(n)}\|$ and $\|\varphi\|_{cb} := \sup_n \|\varphi\|_n$. The map φ is *completely bounded* if $\|\varphi\|_{cb} < +\infty$, *completely contractive* if $\|\varphi\|_{cb} \leq 1$, and *completely isometric* if $\varphi^{(n)}$ is isometric for every $n \in \mathbb{N}$. In the following all the maps are supposed to be linear.

If X is an operator space and $q \in \mathbb{N}$, then $\text{MIN}_q(X)$ is the operator space with same linear structure as X and matrix norms

$$\|x\|_{M_k(\text{MIN}_q(X))} = \sup_{\phi} \left\| \phi^{(k)}(x) \right\|_{M_k(M_q)}$$

for $x \in M_k(X)$, where ϕ ranges over all complete contractions from X to M_q . The space $\text{MIN}_q(X)$ is characterized by the following property [OR, §2]: for any operator space Z and linear map $\varphi : Z \rightarrow X$, one has that

$$\|\varphi : Z \rightarrow \text{MIN}_q(X)\|_{cb} = \|\varphi : Z \rightarrow X\|_q.$$

Following [L1], we say that X is an M_q -space if the identity map $id : \text{MIN}_q(X) \rightarrow X$ is a complete isometry. In an M_q -space X the norms on $M_k(X)$ for $k \neq q$ are completely determined from the norm in $M_q(X)$. Therefore one can regard an M_q -space as a linear space X with a norm on $M_q(X)$. The spaces arising in this way can be abstractly characterized by the analog of Equation (1) where only matrices of size q are considered [L1, Théorème I.1.9]. Furthermore M_q -spaces are precisely the operator spaces that admit completely isometric embedding into $C(K, M_q)$ for some compact Hausdorff space K . Adopting this point of view, *Banach spaces* can be identified with M_1 -spaces. The characterizing property of MIN_q shows that if $\varphi : X \rightarrow Y$ is a linear map between M_q -spaces, then $\|\varphi\|_{cb} = \|\varphi\|_q$. Furthermore the inclusion functor from the category of M_q -spaces to the category of operator spaces determines an equivalence of categories with inverse the functor MIN_q .

A finite-dimensional operator space X is defined to be *1-exact* if for every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and a completely contractive injective map $\varphi : X \rightarrow M_n$ such that $\|\varphi^{-1}\|_{cb} \leq 1 + \varepsilon$ [P2, §17]. (Here and in the following, if φ is an injective not necessarily surjective map, we denote by φ^{-1} the inverse of φ regarded as a map from the range of φ to the domain of φ .) Equivalently X is 1-exact if and only if for every $\varepsilon > 0$ there exists $q \in \mathbb{N}$ such that

$$\|id_X : \text{MIN}_q(X) \rightarrow X\|_{cb} \leq 1 + \varepsilon.$$

An arbitrary operator space X is then 1-exact if every finite-dimensional subspace of X is 1-exact. Clearly any M_q -space is 1-exact, and the 1-exact operator spaces are those that can be locally approximated by M_q -spaces.

An operator space X is nuclear [ER2, §14.6] if for every finite subset A of X and $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and completely contractive maps $\varphi : X \rightarrow M_n$ and $\psi : M_n \rightarrow X$ such that $\|(\psi \circ \varphi)(x) - x\| \leq \varepsilon$ for every $x \in A$. A nuclear operator space is in particular 1-exact [ER2, Corollary 14.6.2]. More details about operator spaces can be found in the monographs [ER2, P2, BLM, P1]. The notion of M_q -space has been introduced and studied in [L1], and used in [OR, O, L2].

The Gurarii operator space \mathbb{NG} has been defined in [O] to be a separable 1-exact operator space satisfying the following approximate injectivity property: whenever $E \subset F$ are finite-dimensional 1-exact operator spaces, $\varphi : E \rightarrow \mathbb{NG}$ is a complete isometry, and $\varepsilon > 0$, then φ can be extended to a linear map $\widehat{\varphi} : F \rightarrow \mathbb{NG}$ such that $\|\widehat{\varphi}\|_{cb} \|\widehat{\varphi}^{-1}\|_{cb} \leq 1 + \varepsilon$. The uniqueness of such a space up to complete isometry has been proved in [L2], where moreover several equivalent characterizations of \mathbb{NG} can be found; see [L2, Proposition 4.8]. It is furthermore proved therein that a separable operator space is 1-exact if and only if it embeds completely isometrically into \mathbb{NG} [L2, Theorem 4.9]. This can be regarded as an operator space analog of Kirchberg's exact embedding theorem [RS, Theorem 6.3.11]. Theorem 2.5 below provides an operator space analog of Kirchberg's nuclear embedding theorem: a

separable operator space X is nuclear if and only if it embeds completely isometrically as a subspace of $\mathbb{N}\mathbb{G}$ that is the range of a completely contractive projection.

2.2. Amalgamation of operator spaces. The proof of the following lemma is similar to the proof of [L2, Lemma 3.1]. We present here the main ideas for convenience of the reader.

Lemma 2.1. Suppose that X, \widehat{X}, Y are M_q -spaces, $k \in \mathbb{N}$, and $\delta \geq 0$. Let $\phi : X \rightarrow \widehat{X}$ and $f : X \rightarrow Y$ be linear maps such that ϕ is injective, and $\|f \circ \phi^{-1}\|_{cb} \leq 1 + \delta$. Then there exist an M_q -space \widehat{Y} , a complete contraction $\widehat{f} : \widehat{X} \rightarrow \widehat{Y}$, and a complete isometry $j : Y \rightarrow \widehat{Y}$ such that $\|j \circ f - \widehat{f} \circ \phi\|_k \leq \delta$. If f is injective and $\|\phi \circ f^{-1}\|_{cb} \leq 1 + \delta$, then \widehat{f} is completely isometric. If \widehat{X}, Y are finite-dimensional, then \widehat{Y} is finite-dimensional. The space \widehat{Y} has the following property. If Z is an M_q -space, and $g : \widehat{X} \rightarrow Z$, $i : Y \rightarrow Z$ are completely contractive maps such that $\|i \circ f - g \circ \phi\|_k \leq \delta$, then there exists a unique linear map $\tau : \widehat{Y} \rightarrow Z$ such that $\tau \circ \widehat{f} = g$ and $\tau \circ j = i$. Moreover τ is completely contractive.

Proof. Let $\widehat{X} \oplus Y$ be the algebraic direct sum of \widehat{X} and Y . Consider the collection \mathcal{F} of linear maps from $\widehat{X} \oplus Y$ to M_q of the form

$$(x, y) \mapsto \theta_X(x) + \theta_Y(y)$$

where $\theta_X : \widehat{X} \rightarrow M_q$ and $\theta_Y : Y \rightarrow M_q$ are completely contractive maps such that $\|\theta_X \circ \phi - \theta_Y \circ f\|_k \leq \delta$. Let \widehat{Y} be the operator space obtained from the vector space $\widehat{X} \oplus Y$ and the collection of linear maps \mathcal{F} as in [BLM, §1.2.16]. Observe that \widehat{Y} is in fact an M_q -space. Let $\widehat{f} : \widehat{X} \rightarrow \widehat{Y}$ and $j : Y \rightarrow \widehat{Y}$ be the linear maps induced by the first and second coordinate inclusions. It follows immediately from the definitions that \widehat{f} and j are completely contractive maps such that $\|\widehat{f} \circ \phi - j \circ f\|_k \leq \delta$. It remains to show that j is a complete isometry. Suppose that $y \in M_q(Y)$ and $\eta : Y \rightarrow M_q$ is a complete contraction such that $\|y\| = \|\eta^{(q)}(y)\|$. Then $\frac{1}{1+\delta}(\eta \circ f \circ \phi^{-1}) : \phi[X] \rightarrow M_q$ is a complete contraction and hence it extends to a complete contraction $\psi : \widehat{X} \rightarrow M_q$ by injectivity of M_q in the category of M_q -spaces and complete contractions; see [L1, Proposition I.1.16]. Observe that $\|\psi \circ \phi - \eta \circ f\|_q \leq \delta$. Therefore we have that

$$\|j^{(q)}(y)\| \geq \|\eta^{(q)}(y)\| \geq \|y\|.$$

This concludes the proof that j is a complete isometry. The proof that \widehat{f} is a complete isometry under the assumption that f is injective and $\|\phi \circ f^{-1}\|_{cb} \leq 1 + \delta$ is entirely analogous. Suppose now that Z is an M_q -space, and $g : \widehat{X} \rightarrow Z$ and $i : Y \rightarrow Z$ are completely contractive maps such

that $\|i \circ f - g \circ \phi\|_k \leq \delta$. Then setting

$$\tau((x, y) + N) = g(x) + i(y)$$

gives a well defined complete contraction $\tau : \widehat{Y} \rightarrow Z$. It is not difficult to verify that τ satisfies the desired conclusions. \square

A minor modification of the above argument provides the following straightforward generalization.

Lemma 2.2. Suppose that $k, N \in \mathbb{N}$, X_n, \widehat{X}_n, Y for $n \leq N$ are finite-dimensional M_q -spaces, $k \leq q$, and $\delta_n \geq 0$. Suppose that $\phi_n : X_n \rightarrow \widehat{X}_n$ and $f_n : X_n \rightarrow Y$ are linear maps such that

$$\max \{ \|f_n \circ \phi_n^{-1}\|_k, \|\phi_n \circ f_n^{-1}\|_k \} \leq 1 + \delta_n$$

for every $n \leq N$. There exists a finite-dimensional M_q -space \widehat{Y} , a complete isometry $j : Y \rightarrow \widehat{Y}$ and complete isometries $\widehat{f}_n : \widehat{X}_n \rightarrow Y$ such that $\|\widehat{f}_n \circ \phi_n - j \circ f_n\|_k \leq \delta_n$ for every $n \leq N$. Moreover if Z is an M_q -space, $g_n : \widehat{X}_n \rightarrow Z$ and $i : Y \rightarrow Z$ are completely contractive maps such that $\|g_n \circ \phi_n - i \circ f_n\|_k \leq \delta_n$, then there exists a unique linear map $\tau : \widehat{Y} \rightarrow Z$ such that $\tau \circ j = i$ and $\tau \circ \widehat{f}_n = g_n$. Furthermore τ is completely contractive.

We will refer to the space \widehat{Y} constructed in Lemma 2.2 as the *k-amalgamated coproduct* over Y of the maps $f_n : X_n \rightarrow Y_n$ and $\phi_n : X_n \rightarrow Y$ with tolerance δ_n for $n \leq N$.

2.3. A characterization of nuclearity. In the proof of the main theorem we will need the following characterization of nuclearity for operator spaces.

Lemma 2.3. Suppose that Z is a 1-exact operator space. The following statements are equivalent:

- (1) For every finite-dimensional operator spaces E, F , $\delta > 0$, linear maps $\phi : E \rightarrow F$ and $f : E \rightarrow Z$ such that ϕ is injective and $\|f \circ \phi^{-1}\|_{cb} < 1 + \delta$, there exists a completely contractive map $g : F \rightarrow Z$ such that $\|g|_E - f\| < \delta$;
- (2) For every $q \in \mathbb{N}$, subspace $E \subset M_q$, completely contractive linear map $f : E \rightarrow Z$, and $\varepsilon > 0$, there exists a completely contractive map $g : M_q \rightarrow Z$ such that $\|g \circ \phi - f\| < \varepsilon$;
- (3) Z is nuclear.

Proof. The argument is straightforward. For convenience of the reader, we sketch below the proofs of the nontrivial implications.

(2) \Rightarrow (3): Suppose that \bar{a} is a tuple in Z and $\varepsilon > 0$. Fix $\delta > 0$ small enough. Pick $q \in \mathbb{N}$, a subspace E of M_q and a completely contractive linear map $f : E \rightarrow Z$ with range $\text{span}(\bar{a})$ such that $\|f^{-1}\|_{cb} < 1 + \delta$. By assumption there exists a completely contractive map $\rho : M_q \rightarrow Z$ such that $\|\rho - f\|_{cb} < \delta$. Observe now

that $\gamma := \frac{1}{1+\delta}f^{-1} : \text{span}(\bar{a}) \rightarrow M_q$ is completely contractive and $\|\rho \circ \gamma(a_i) - a_i\| < \varepsilon$ for δ small enough.

(3) \Rightarrow (1): Suppose that E, F are finite-dimensional 1-exact operator spaces, $\delta > 0$, and $n \in \mathbb{N}$. Fix linear maps $\phi : E \rightarrow F$ and $f : E \rightarrow Z$ such that ϕ is injective. Set $\|f \circ \phi^{-1}\|_{cb} = 1 + \eta$ and let ε be a strictly positive real number. Since Z is nuclear there exist $q \in \mathbb{N}$ and completely contractive maps $\rho : Z \rightarrow M_q$ and $\gamma : M_q \rightarrow Z$ such that $\|\gamma \circ \rho \circ f - f\| < \varepsilon$. By injectivity of M_k there exists a completely contractive map $h : F \rightarrow M_k$ that extends $\frac{1}{1+\eta}(\rho \circ f \circ \phi^{-1})$. Let then g be $\gamma \circ h$ and observe that $\|g \circ \phi - f\| < \eta + \varepsilon$.

□

2.4. The nuclear $\mathbb{N}\mathbb{G}$ -embedding theorem . The main result of this section is Theorem 2.5, characterizing separable nuclear operator spaces as the ranges of completely contractive projections of the Gurarij operator space. The proof is inspired by a characterization of retracts of certain Fraïssé limits due to Dolinka; see [D, Theorem 3.2]. Dolinka's result has later been extended by Kubiś in [K2] to more general Fraïssé limits. The forward implication in Theorem 2.5 has already been obtained by Oikhberg [O, Theorem 1.1] under the stronger assumption that the operator space X is an $\mathcal{OL}_{\infty,1+}$ space as defined in [ER1].

Fix a sequence (ε_n) of strictly positive real numbers such that $\varepsilon_n \leq 2^{-2n}$ for every $n \in \mathbb{N}$. We say that a subset D of a metric space X is ε -dense for some $\varepsilon > 0$ if every element of X is at distance at most ε from some element of D . For $m \in \mathbb{N}$ let D_m be a finite ε_m -dense subset of the unit ball of M_m and let $(\bar{a}_{m,i})$ be an enumeration of the finite tuples from D_m . Set $E_{m,i} = \langle \bar{a}_{m,i} \rangle \subset M_m$ for $i, m \in \mathbb{N}$.

Assume that (Z_n) is a direct system of finite-dimensional 1-exact operator spaces with completely isometric connective maps $j_n : Z_n \rightarrow Z_{n+1}$. Suppose that $D_n^Z \subset \text{Ball}(Z_n)$ is an ε_n -dense finite subset of Z_n . Define Z to be the corresponding direct limit, and identify Z_n with its image inside Z .

Lemma 2.4. Suppose that for any $i, m \in \mathbb{N}$ with $i, m \leq n$ and complete contraction $f : E_{m,i} \rightarrow Z_n$ such that $f(\bar{a}_{m,i}) \subset D_n^Z$ there exists a complete isometry $\hat{f} : M_m \rightarrow Z_{n+1}$ such that $\|\hat{f}|_E - j_n \circ f\| \leq \|f^{-1}\|_{cb} - 1 + \varepsilon_n$. Then Z is completely isometric to $\mathbb{N}\mathbb{G}$.

Proof. Observe that Z is separable and 1-exact. By [L2, Proposition 4.8] it is enough to show that if $E \subset M_m$, $g : E \rightarrow Z$ is a complete isometry, and $\delta > 0$, then there exists a complete isometry $\hat{g} : M_m \rightarrow Z$ such that $\|\hat{g}|_E - g\| \leq \delta$. The “small perturbation argument”—see [BO, Lemma 12.3.15]—shows that, for such a g and $\delta > 0$, there exist $n \geq i, m$ and a complete contraction $f : E_{m,i} \rightarrow Z_n$ such that $\varepsilon_n \leq \delta$, $f(\bar{a}_{m,i}) \subset D_n^Z$, $\|\hat{f}^{-1}\|_{cb} \leq 1 + \delta$, and $\|f - g\| \leq \delta$. The conclusion then follows easily by applying the hypothesis to f . □

Theorem 2.5. Suppose that X is a separable 1-exact operator space. Then X is nuclear if and only if there exist a complete isometry $\phi : X \rightarrow \mathbb{N}\mathbb{G}$ and a completely contractive projection of $\mathbb{N}\mathbb{G}$ onto the range of ϕ .

Proof. It follows easily from the characterizing property of $\mathbb{N}\mathbb{G}$ and Lemma 2.3 that $\mathbb{N}\mathbb{G}$ is nuclear. Therefore the range of a completely contractive projection of $\mathbb{N}\mathbb{G}$ is nuclear as well. Conversely, suppose that X is a nuclear. Our goal is to construct an operator space Z completely isometric to $\mathbb{N}\mathbb{G}$, a completely isometric embedding $\eta : X \rightarrow Z$, and a completely contractive map $\pi : Z \rightarrow X$ such that $\pi \circ \eta = id_X$. We will define by recursion on n

- an increasing sequence (q_n) in \mathbb{N} such that $q_n \geq n$,
- an increasing sequence (X_n) of finite-dimensional subspaces X with inclusion maps $i_n : X_n \rightarrow X_{n+1}$ such that $\bigcup_n X_n$ is dense in X and

$$\|id_{X_n} : \text{MIN}_{q_n}(X_n) \rightarrow X_n\|_{cb} \leq 1 + \varepsilon_n,$$

- an inductive sequence (Z_n) of finite-dimensional 1-exact operator spaces with completely isometric connective maps $j_n : Z_n \rightarrow Z_{n+1}$ such that Z_n is an M_{q_n} -space,
- completely isometric maps $\eta_n : \text{MIN}_{q_n}(X_n) \rightarrow Z_n$ such that

$$\|j_n \circ \eta_n - \eta_{n+1} \circ i_n\| \leq \varepsilon_n,$$

- complete contractions $\pi_n : Z_n \rightarrow X_n$ such that

$$\|\pi_{n+1} \circ j_n - i_n \circ \pi_n\| \leq \varepsilon_n \text{ and } \|\pi_n \circ \eta_n - id_{X_n}\| \leq \varepsilon_n,$$

- finite ε_n -dense subset D_n^X of $\text{Ball}(X_n)$ and D_n^Z of $\text{Ball}(Z_n)$ such that $i_n[D_n^X] \subset D_{n+1}^X$, $j_n[D_n^Z] \subset D_{n+1}^Z$, and $\pi_n[D_n^Z] \subset D_n^X$,

such that

- (1) For every $i, m \leq n$ and $f : E_{m,i} \rightarrow X_n$ complete contraction such that $f(\bar{a}_{m,i}) \subset D_n^X$ there exists a complete contraction $\hat{f} : M_m \rightarrow X_{n+1}$ such that $\|\hat{f}|_E - i_n \circ f\| \leq \varepsilon_n$;
- (2) For every $i, m \leq n$ and $f : E_{m,i} \rightarrow Z_n$ complete contraction such that $f(\bar{a}_{m,i}) \subset D_n^Z$ there exists a complete isometry $\hat{f} : M_m \rightarrow Z_{n+1}$ such that $\|\hat{f}|_E - j_n \circ f\| \leq \|f^{-1}\|_{cb} - 1 + \varepsilon_n$.

Fix a dense sequence (w_n) in X . For $k = 1$, let $X_1 = \text{span}\{w_1\}$, q_1 large enough such that $\|id_{X_1} : \text{MIN}_{q_1}(X_1) \rightarrow X_1\|_{cb} \leq 1 + \varepsilon_1$, and define $Z_1 = \text{MIN}_{q_1}(X_1)$. Suppose that $X_k, Z_k, \pi_k, \eta_{k-1}, j_{k-1}$ have been defined for $k \leq n$. Observe that Condition (1) above concerns only finitely many functions f . Therefore one can find a subspace X_{n+1} of X containing $X_n \cup \{w_{n+1}\}$ and satisfying Condition (1) by applying finitely many times Proposition 2.3 and the fact that X is nuclear. Pick $q_{n+1} \geq \max\{n+1, q_n\}$ such that

$$\|id : \text{MIN}_{q_{n+1}}(X_{n+1}) \rightarrow X_{n+1}\|_{cb} \leq 1 + \varepsilon_{n+1}.$$

Let Z_{n+1} be the 1-amalgamated coproduct of $M_{q_{n+1}}$ -spaces over Z_n (see Lemma 2.2) of the maps $\eta_n : \text{MIN}_{q_n}(X_n) \rightarrow Z_n$ and $i_n : \text{MIN}_{q_n}(X_n) \rightarrow$

$\text{MIN}_{q_{n+1}}(X_n)$ with tolerance ε_n and of the maps $f : E_{m,i} \rightarrow Z_n$ and $E_{m,i} \hookrightarrow M_m$ (inclusion map) with tolerance $\|f^{-1}\|_{cb} - 1 + \varepsilon_n$, where m, i range in $\{1, \dots, n\}$ and f range among all the (finitely many) injective complete contractions such that $f(\bar{a}_{m,i}) \subset D_n^Z$. Let $j_n : Z_n \rightarrow Z_{n+1}$ and $\eta_{n+1} : \text{MIN}_{q_{n+1}}(X_{n+1}) \rightarrow Z_{n+1}$ be the canonical complete isometries such that $\|j_n \circ \eta_n - \eta_{n+1} \circ i_n\| \leq \varepsilon_n$. It remains to define the complete contraction $\pi_{n+1} : Z_{n+1} \rightarrow X_{n+1}$. Observe that $i_n \circ \pi_n : Z_n \rightarrow X_{n+1}$ is a complete contraction such that

$$\|i_n \circ \pi_n \circ \eta_n - i_n\| \leq \varepsilon_n.$$

Furthermore for every $i, m \leq n$ and for any complete contraction $f : E_{m,i} \rightarrow Z_n$ such that $f(\bar{a}_{m,i}) \subset D_n^Z$, $\pi_n \circ f : E_{m,i} \rightarrow X_n$ is a complete contraction such that $(\pi_n \circ f)(\bar{a}_{m,i}) \subset D_n^X$. Therefore by Condition (1) there exists a complete contraction $g : M_m \rightarrow X_{n+1}$ such that

$$\left\| g|_{E_{m,i}} - (i_n \circ \pi_n) \circ f \right\| \leq \varepsilon_n.$$

It follows from the universal property of the 1-amalgamated coproduct that there exists a complete contraction $\pi : Z_{n+1} \rightarrow \text{MIN}_{q_{n+1}}(X_{n+1})$ such that $\eta_{n+1} \circ \pi = id_{X_{n+1}}$ and $\pi \circ j_n = i_n \circ \pi_n$. Define then $\pi_{n+1} = \frac{1}{1+\varepsilon_{n+1}}\pi$. This concludes the recursive construction. We now define Z to be the direct limit $\lim_{(\phi_n)} Z_n$, $\pi := \lim_n \pi_n$, and $\eta := \lim_n \eta_n$. It is immediate to verify that the conditions above ensure that π and η satisfy the desired requirements. Furthermore Z is completely isometric to \mathbb{NG} by Lemma 2.4. \square

3. NUCLEAR OPERATOR SYSTEMS

3.1. Some background notions about operator systems. An *operator system* is a closed subspace X of $B(H)$ that contains the identity operator (the *unit*) and it is closed by taking adjoints. The inclusion $X \subset B(H)$ and the identification of $M_n(X)$ with a subspace of $B(H^{\oplus n})$ defines *positive cones* on all the matrix amplifications of X . These are induced by the notion of positivity in $B(H)$. (An operator $T \in B(H)$ is positive provided that $\langle T\xi, \xi \rangle \geq 0$ for any $\xi \in H$.) The Choi-Effros characterization [CE, Theorem 4.4] provides an abstract characterization of operator systems among the *matricially ordered* vector spaces endowed with a linear involution (the adjoint map) and a distinguished element (the unit). An operator system is in particular an operator space, and an abstract characterization of operator systems among operator spaces with a distinguished element (the unit) has been given in [BN].

A map φ between operator systems is *unital* if it maps the unit to the unit, *positive* if it maps positive elements to positive elements, and *completely positive* if all its amplifications are positive. A fundamental fact about operator spaces is that the matricial order structure determines the matricial norms, and vice versa. This implies that a unital linear map between operator systems is completely positive if and only if it is completely

contractive. A unital completely positive linear map will be in the following simply called a *ucp map*. A surjective ucp map with ucp inverse is called a *complete order isomorphism*. A *complete order embedding* is a complete order isomorphism onto its image. Observe that a complete order embedding is in particular a complete isometry.

The minimal operator system structure $\text{OMIN}_q(X)$ on an operator system X has been introduced and studied in [X]. This is defined similarly as $\text{MIN}_q(X)$ by only considering ucp maps. It is shown in [X] that $\text{OMIN}_q(X)$ has analogous properties as $\text{MIN}_q(X)$ as long as one replaces (completely) contractive linear maps with (completely) positive unital linear maps; see [L3, §2.7]. An M_q -system will then be an operator system X such that the identity map $\text{id} : \text{OMIN}(X) \rightarrow X$ is a complete order isomorphism or, equivalently, X admits a complete order embedding in $C(K, M_q)$ for some compact Hausdorff space K . The notions of 1-exact and nuclear operator systems can be defined as for operator spaces by replacing complete contractions with ucp maps. It turns out that an operator system is 1-exact or nuclear, respectively, if and only if it is 1-exact or nuclear as an operator space; see [HP, Theorem 3.5] and [KPTT, Proposition 5.5].

3.2. The nuclear GS-embedding theorem . The approximate amalgamation results for operator spaces from Subsection 2.2 carry over with little change to the operator systems case. The only extra ingredient needed is [L3, Lemma 3.3].

Lemma 3.1. Suppose that $N \in \mathbb{N}$, X_n, \widehat{X}_n, Y for $n \leq N$ are finite-dimensional M_q -systems, $k \leq q$, and $\delta_n \in [0, 1]$ for $n \leq N$. Suppose that $\phi_n : X_n \rightarrow \widehat{X}_n$ and $f_n : X_n \rightarrow Y$ linear maps such that

$$\max \{ \|f_n \circ \phi_n^{-1}\|_k, \|\phi_n \circ f_n^{-1}\|_k \} \leq 1 + \delta_n$$

for every $n \leq N$. There exists a finite-dimensional M_q -system \widehat{Y} , a complete isometry $j : Y \rightarrow \widehat{Y}$ and complete isometries $\widehat{f}_n : \widehat{X}_n \rightarrow Y$ such that $\|\widehat{f}_n \circ \phi_n - j \circ f_n\|_k \leq 100 \dim(X_n) \delta_n^{\frac{1}{2}}$. Moreover if Z is an M_q -system, $g_n : \widehat{X}_n \rightarrow Z$ and $i : Y \rightarrow Z$ are ucp maps such that $\|g_n \circ \phi_n - i \circ f_n\|_k \leq 100 \dim(X_n) \delta_n^{\frac{1}{2}}$, then there exists a unique linear map $\tau : \widehat{Y} \rightarrow Z$ such that $\tau \circ j = i$ and $\tau \circ \widehat{f}_n = g_n$. Furthermore τ is ucp.

The following characterization of nuclearity for operator systems can then be proved similarly as 2.3. Again one needs to use [L3, Lemma 3.3].

Lemma 3.2. Suppose that Z is a 1-exact operator system. The following statements are equivalent:

- (1) For every finite-dimensional operator systems E, F , $\delta > 0$, unital linear maps $\phi : E \rightarrow F$ and $f : E \rightarrow Z$ such that ϕ is injective and $\|f \circ \phi^{-1}\|_{cb} < 1 + \delta$, there is a ucp map $g : F \rightarrow Z$ such that $\|g|_E - f\| < 100 \dim(E) \delta^{\frac{1}{2}}$;

- (2) For every $q \in \mathbb{N}$, $E \subset M_q$ subsystem, ucp map $f : E \rightarrow Z$, and $\varepsilon > 0$, there exists a ucp map $g : F \rightarrow Z$ such that $\|g \circ \phi - f\| < \varepsilon$;
- (3) Z is nuclear.

The operator system analog of Kirchberg's nuclear embedding theorem can also be obtained with similar methods as Theorem 2.5.

Theorem 3.3. Suppose that X is a separable 1-exact operator system. Then X is nuclear if and only if there exist a complete order embedding $\phi : X \rightarrow \mathbb{GS}$ and a unital completely positive projection of \mathbb{GS} onto the range of ϕ .

In order to prove Theorem 3.3 one can proceed as in the proof of Theorem 2.5, by replacing [L2, Proposition 4.8] with the characterization of \mathbb{GS} given by [L3, Proposition 4.2]. All linear maps should be replaced by unital linear maps. In this setting the functor OMIN_q plays the role of MIN_q .

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